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Dissecting d-Cubes into Smaller d-Cubes

Matthew Hudelson

Department of Pure and Applied Mathematics, Washington State University, Pullman, Washington 99164-3113

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In this paper, we explore the following question: Given integers d and k, is it possible to subdivide a d-dimensional cube into k smaller d-dimensional cubes? In particular, we investigate bounds on the integer $c(d)$ which is the smallest integer [illegible] given k, the asymptotic behavior of $c(d)$ for those d such that $\gcd(2^d-1,$ $k^d-1)=1$. Specifically, we show that if $\gcd(2^d-1, 3^d-1)$ then $c(d)<6^d$ and that if $\gcd(2^d-1, k^d-1)$ then $c(d)=O((2k)^d)$. Finally, we derive the general asymptotic bound $c(d)=O((2d)^{d-1})$ which improves the currently known bound of $c(d)=O((2d)^d)$. © 1998 Academic Press, Inc.

1. BACKGROUND

Croft, Falconer, and Guy [1] discuss the following problem which has been attributed to Hadwiger and proposed by Fine and Niven [3]: *Given d, for which integers k can the unit d-dimensional cube Q_d be decomposed into k smaller d-cubes*? If we let D_d represent the set of all such integers, then let $c(d)$ be the smallest integer such that if $k \geqslant c(d)$ then $k \in D_d$.

In order to investigate $c(d)$ further, we prove the following elementary result.

THEOREM 1. *Suppose A is a set of positive integers $\{a_1, ..., a_{2^d-1}\}$ such that, for each k, Q^d can be subdivided into a_k d-cubes and such that for each $m \in \{0, ..., 2^d-2\}$ there is a k_m such that $a_{k_m} \equiv m \bmod 2^d-1$. Then $c(d) \leqslant \max(A)-2^d+2$.*

By definition, we cannot subdivide Q_d into $c(d)-1$ cubes. Suppose $c(d)>\max(A)-2^d+2$. There exists some $a \in A$ such that $a \equiv c(d)-1 \bmod 2^d-1$, so we can write $c(d)-1=p(2^d-1)+a$ for some integer p. If $p=0$, then we can subdivide Q_d into $a=c(d)-1$ cubes which is a contradiction. If $p>0$, then we subdivide Q_d into a cubes, then subdivide one

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of the smaller cubes into 2^d still smaller cubes, and then repeat this latter step $p-1$ more times to yield a subdivision of Q_d into $c(d)-1$ cubes which is again a contradiction. Therefore, $p<0$. It follows that

$$c(d)-1=p(2^d-1)+a \leqslant p(2^d-1)+\max(A) \leqslant \max(A)-(2^d-1)$$

and so $c(d) \leqslant \max(A)-2^d+2$, which contradicts our original assumption. The result follows. ▮

2. BOUNDS ON $c(d)$ FOR $d \leqslant 5$

2.1. *The Cases when $d \leqslant 3$*

Since it is possible to subdivide a line segment into any number of smaller line segments, $c(1)=1$, and in fact, $D_1=\mathbf{N}$. In Fig. 1, we demonstrate that $c(2) \leqslant 6$ by showing how to subdivide a square into one, six, and eight smaller squares, respectively. In this case, we have $A=\{6, 7, 8\}$ although $A=\{1, 6, 8\}$ or $A=\{4, 6, 8\}$ would also serve to show that $c(2) \leqslant 6$. It is not possible to subdivide a square into two, three, or five smaller squares, so, in fact, $c(2)=6$ and $D_2=\mathbf{N} \backslash \{2, 3, 5\}$.

Exact values of $c(d)$ are unknown for $d>2$, although it is conjectured that $c(3)=48$. To see that $c(3) \leqslant 48$, we show that $A=\{1, 20, 38, 39, 49, 51, 54\}$ is a subset of D_3. Then Theorem 1 gives us $c(3) \leqslant 54-8+2=48$. Figures 2 and 3 show how to subdivide a cube into a smaller cubes for each $a \in A$.

The subdivision of a cube into 54 smaller cubes is the most intricate and was discovered by Rychener and Zbinden (see [1]).

2.2. *The Four-Dimensional Case*

We show that $c(4) \leqslant 809$ by showing

$$\{1, 66, 131, 370, 435, 500, 634, 672, 693, 699, 737, 758, 764, 802, 823\} \subset D_4.$$

To show this, we first note that subdividing a cube into 3^4 cubes and then coalescing 2^4 of these into a cube results in a subdivision of a cube

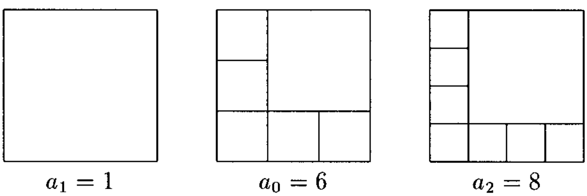


Fig. 1. Subdividing a square into smaller squares, $a \in \{1, 6, 8\}$.

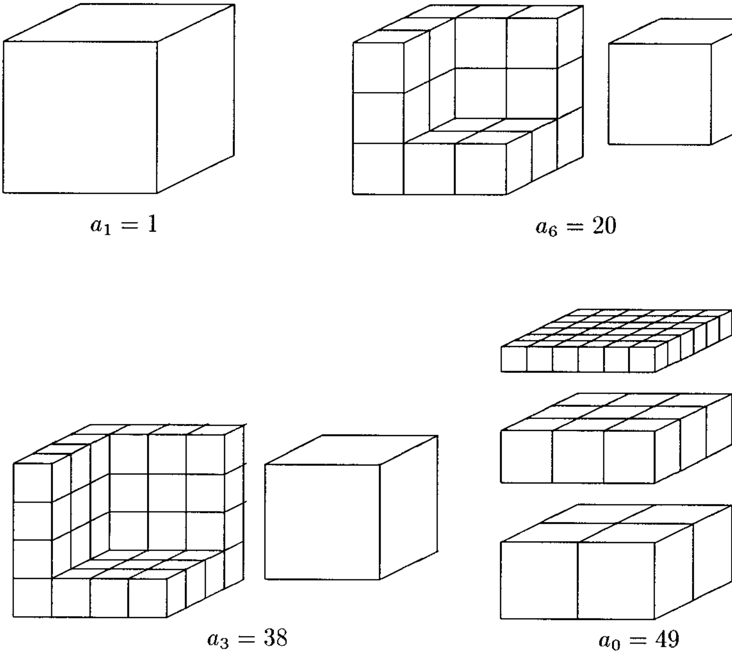


Fig. 2. Subdividing a cube into smaller cubes, $a \in \{1, 20, 38, 49\}$.

into 66 smaller cubes. Moreover, this strategy can be used to show that if $k \in D_4$, then $k + 65 \in D_4$. With this in mind, it suffices to show that $\{1, 370, 634, 672, 693\} \subset D_4$. Clearly, $a_1 = 1 \in D_4$ as we can leave the cube in one piece. Also, it is easy to see that $a_0 = 370 = 5^4 - 4^4 + 1 \in D_4$ and $a_2 = 672 = 6^4 - 5^4 + 1 \in D_4$.

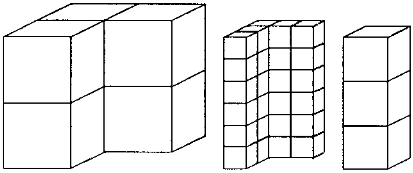
Next, we note that $10^4 = 14 \cdot 5^4 + 2 \cdot 4^4 + 8 \cdot 2^4 + 610 \cdot 1^4$, and if we can assemble these cubes into one cube, the result is a subdivision of Q_4 into $a_4 = 634$ smaller cubes. Table I contains a list of "least corners" (a, b, c, d) for the "k^4" hypercubes, $k \geqslant 2$, in the subdivision. The other corners of each hypercube are $(a + a', b + b', c + c', d + d')$, a', b', c', and d' each in $\{0, k\}$.

Finally, to show $a_3 = 693 \in D_4$, we note that $10^4 = 13 \cdot 5^4 + 5 \cdot 3^4 + 53 \cdot 2^4 + 622 \cdot 1^4$. The corresponding list of least corners is contained in Table II.

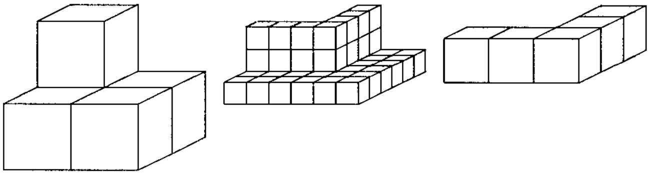
2.3. *An Upper Bound for* $c(5)$

At this point, we demonstrate that $c(5) \leqslant 1891$, a bound credited to William Scott by Croft *et al.* [1, page 85]. We reproduce this bound to introduce a strategy which we will generalize in Section 3.

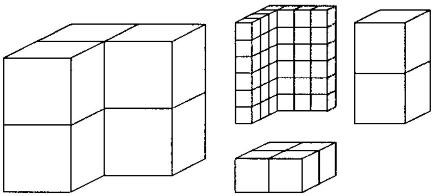
Our basic strategy will be, given $i \in \{0, 1, ..., 30\}$, to subdivide Q_5 into 32 cubes with half the edge length (1/2-cubes) and then to subdivide i of



$a_4 = 39$



$a_2 = 51$



$a_5 = 54$

Fig. 3. Subdividing a cube into smaller cubes, $a \in \{39, 51, 54\}$.

these each into $3^5 = 243$ 1/6-cubes. As a result, we will have subdivided Q_d into $(3^5-1)i+(2^5)$ smaller cubes. Since $\gcd(2^5-1, 3^5-1)=1$, it follows that distinct choices i yield distinct choices of j such that $(3^5-1)i \equiv j \bmod (2^5-1)$. Furthermore, if we make a judicious choice of which i of the 1/2-cubes to further subdivide, we will be able to recoalesce a number of groups of 2^5 of the 1/6-cubes into 1/3-cubes.

TABLE I

Least Corners for the $a_4 = 634$ Case

"5^4" hypercubes:	(0, 0, 0, 0)	(0, 0, 0, 5)	(0, 0, 5, 0)	(0, 0, 5, 5)
	(0, 5, 0, 0)	(0, 5, 0, 5)	(0, 5, 5, 0)	(0, 5, 5, 5)
	(5, 0, 0, 0)	(5, 0, 0, 5)	(5, 0, 5, 0)	(5, 0, 5, 5)
	(5, 5, 0, 0)	(5, 5, 0, 5)		
"4^4" hypercubes:	(5, 5, 5, 0)	(5, 5, 5, 4)		
"2^4" hypercubes:	(5, 5, 5, 8)	(5, 5, 7, 8)	(5, 7, 5, 8)	(5, 7, 7, 8)
	(7, 5, 5, 8)	(7, 5, 7, 8)	(7, 7, 5, 8)	(7, 7, 7, 8)

TABLE II

Least Corners for the $a_3 = 693$ Case

"5^4" hypercubes:	(0, 0, 0, 0)	(0, 0, 0, 5)	(0, 0, 5, 0)	(0, 0, 5, 5)
	(0, 5, 0, 0)	(0, 5, 0, 5)	(0, 5, 5, 0)	(0, 5, 5, 5)
	(5, 0, 0, 0)	(5, 0, 0, 5)	(5, 0, 5, 0)	(5, 0, 5, 5)
	(5, 5, 0, 0)			
"3^4" hypercubes:	(5, 5, 7, 0)	(5, 5, 7, 3)	(5, 5, 7, 6)	(5, 5, 0, 7)
	(5, 5, 3, 7)			
"2^4" hypercubes:	(5, 5, 0, 5)	(5, 7, 0, 5)	(7, 5, 0, 5)	(7, 7, 0, 5)
	(5, 5, 2, 5)	(5, 7, 2, 5)	(7, 5, 2, 5)	(7, 7, 2, 5)
	(5, 5, 4, 5)	(5, 7, 4, 5)	(7, 5, 4, 5)	(7, 7, 4, 5)
	(5, 5, 5, 0)	(5, 7, 5, 0)	(7, 5, 5, 0)	(7, 7, 5, 0)
	(5, 5, 5, 2)	(5, 7, 5, 2)	(7, 5, 5, 2)	(7, 7, 5, 2)
	(8, 5, 0, 7)	(8, 7, 0, 7)	(5, 8, 0, 7)	(8, 5, 2, 7)
	(8, 7, 2, 7)	(5, 8, 2, 7)	(8, 5, 4, 7)	(8, 7, 4, 7)
	(5, 8, 4, 7)	(8, 5, 8, 0)	(8, 7, 8, 0)	(5, 8, 8, 0)
	(8, 5, 8, 2)	(8, 7, 8, 2)	(5, 8, 8, 2)	(8, 5, 6, 4)
	(8, 7, 6, 4)	(5, 8, 6, 4)	(8, 5, 8, 4)	(8, 7, 8, 4)
	(5, 8, 8, 4)	(8, 5, 6, 6)	(8, 7, 6, 6)	(5, 8, 6, 6)
	(8, 5, 8, 6)	(8, 7, 8, 6)	(5, 8, 8, 6)	(8, 5, 6, 8)
	(8, 7, 6, 8)	(5, 8, 6, 8)	(8, 5, 8, 8)	(8, 7, 8, 8)
	(5, 8, 8, 8)			

Suppose $i = 2^k$. Then we subdivide a $2^k \times 1^{5-k}$ slab of 1/2-cubes into 1/6-cubes and recoalesce $6^k 2^{5-k}$ of these into 1/3-cubes. Therefore, we have subdivided Q_d into $(2^5 - 2^k)$ 1/2-cubes, 3^k 1/3-cubes, and $6^k(3^{5-k} - 2^{5-k})$ 1/6-cubes.

If $i = \sum_p 2^{k_p}$, where each k_p is a distinct nonnegative integer, then for each p, we subdivide a $2^{k_p} \times 1^{5-k_p}$ slab of 1/2-cubes into 1/6 cubes and recoalesce each slab as in the previous paragraph. But we can also recoalesce across slab boundaries to further reduce the total number of cubes into which Q_d is subdivided. In Section 3, we will generalize this recoalescence argument. There, we introduce the function

$$\psi\left(\sum_p 2^{k_p}\right) = \sum_p 2^p 3^{k_p}$$

where $k_0 > k_1 > \cdots > k_m \geqslant 0$. By using the more general results in Section 3, we see that the total number of each type of cube is $(2^5 - i)$ 1/2-cubes, $\psi(i)$ 1/3-cubes, and $(3^5 i - 2^5 \psi(i))$ 1/6-cubes. Thus, the total number of cubes is $\tau(i) = (3^5 - 1)\, i - (2^5 - 1)\, \psi(i) + 2^5$. Table III contains the values of $\psi(i)$ and $\tau(i)$ for $0 \leqslant i \leqslant 30$.

The largest value for $\tau(i)$ in Table III is 1921, so by Theorem 1, $c(5) \leqslant 1921 - 32 + 2 = 1891$.

TABLE III

Values of $\psi(i)$ and $\tau(i)$ for $0 \leqslant i \leqslant 30$

i	$\psi(i)$	$\tau(i)$	i	$\psi(i)$	$\tau(i)$	i	$\psi(i)$	$\tau(i)$
0	0	32	11	37	1547	22	111	1915
1	1	243	12	45	1541	23	119	1909
2	3	423	13	49	1659	24	135	1655
3	5	603	14	57	1653	25	139	1773
4	9	721	15	65	1647	26	147	1767
5	11	901	16	81	1393	27	155	1761
6	15	1019	17	83	1573	28	171	1507
7	19	1137	18	87	1691	29	179	1501
8	27	1131	19	91	1809	30	195	1247
9	29	1311	20	99	1803			
10	33	1429	21	103	1921			

3. ASYMPTOTIC UPPER BOUNDS FOR $c(d)$

3.1. *The Case when* $\gcd(2^d-1, 3^d-1)=1$

We generalize the construction for $d=5$ to the general case when $\gcd(2^d-1, 3^d-1)=1$. For such a d, we define each slab S_k, $k \in \{0, 1, ..., d-1\}$, by

$$S_k = [0, 6)^k \times [0, 3) \times [3, 6)^{d-k-1}.$$

We make our cubes and slabs half open on the higher side to make arguments concerning disjoint unions easier to formulate. We note that S_k is then the disjoint union of 2^k half-open cubes, each three units on a side. We also note that if $j<k$, then $S_j \cap S_k$ is empty since the $k+1$ term of S_j is $[3, 6)$ while the $k+1$ term of S_k is $[0, 3)$.

Next, we define

$$R_k = [0, 6)^k \times [0, 2) \times [4, 6)^{d-k-1}.$$

As an observation, $R_k \subset S_k$ for each k. Next, given a decreasing set of nonnegative integers $X = \{k_0, k_1, ..., k_q\}$, we define N_X to be the cartesian product expansion of R_{k_q} with the (k_0+1), (k_1+1), and so on to the $(k_{q-1}+1)$ terms replaced with $[0, 4)$. If $X=\{k_0\}$, then $N_X = R_{k_0}$. Here, we note that N_X is the disjoint union of $3^k 2^q$ half-open cubes, each two units on a side.

Finally, we define, for the same set X as before, $X_i = \{k_0, ..., k_i\}$, and

$$C_X = \bigsqcup_{i=0}^{p} N_{X_i},$$

where $\bigsqcup$ denotes a disjoint union. We have a disjoint union since if $i > j$, then the $(k_i + 1)$ term of N_{X_i} is $[4, 6)$, while the $(k_i + 1)$ term of N_{X_j} is $[0, 4)$.

We now claim that $C_X \subset \bigsqcup k \in XS_k$. If $x \in C_X$, then $x = (x_1, ..., x_d) \in N_{X_m}$ for some m. Then

$$x_i \in \begin{cases} [0, 6) & \text{if } i \leqslant k_m \\ [0, 2) & \text{if } i = k_m + 1 \\ [0, 4) & \text{if } i \in \{k_0 + 1, k_1 + 1, ..., k_{m\text{-}1} + 1\} \\ [4, 6) & \text{otherwise.} \end{cases}$$

If there is no $k_i \in X_{m-1}$ such that $x_{k_i + 1} \in [0, 3)$, then $x \in S_{k_m}$ and we are done. Otherwise, let k_q be the smallest element of X_{m-1} such that $x_{k_q + 1} \in [0, 3)$. We then have that

$$x_i \in \begin{cases} [0, 6) & \text{if } i \leqslant k_q \\ [0, 3) & \text{if } i = k_q + 1 \\ [3, 6) & \text{otherwise,} \end{cases}$$

and so $x \in S_{m_q}$ and we are done with the claim.

Now, if $\gcd(2^d - 1, 3^d - 1) = 1$, for each $m \in \{0, 1, ..., 2^d - 2\}$, $(3^d - 1)m$ is distinct modulo $2^d - 1$. Therefore, we can subdivide Q_d into 2^d smaller cubes, and subdivide m of these smaller cubes into 3^d still smaller cubes to arrive at a subdivision of Q_d into $2^d + m(3^d - 1)$ cubes.

We improve this simple dissection by using slabs and recoalescing into larger cubes. Suppose $m = \sum_{i=0}^{p} 2^{k_i}$ with the k_i's decreasing. Then we dissect $6[0, 1)^d$ into 2^d three-unit cubes, but we recoalesce the appropriate three-unit cubes into slabs $S_{k_0}, S_{k_1}, ..., S_{k_p}$, leaving $2^d - m$ three-unit cubes. We then dissect each of these slabs into 3^d one-unit cubes.

Above, we showed that, for $X_i = \{k_0, ..., k_i\}$ and $X = X_p$, $C_X \subset \bigsqcup_{k \in X} S_k$ and that $C_X = \bigsqcup_i N_{X_i}$. Furthermore, we saw that N_{X_i} is the disjoint union of $3^{k_i} 2^i$ two-unit cubes. Therefore, C_X is the disjoint union of $\sum_{i=0}^{p} 3^{k_i} 2^i$ two-unit cubes. We call this sum $\psi(m)$ to reflect its dependance on m.

Then, we can recoalesce $2^d \psi(m)$ one-unit cubes of $\bigsqcup_{k \in X} S_k$ into two-unit cubes. As a result, we have dissected $6Q_d$ into $(2^d - m)$ three-unit cubes, $\psi(m)$ two-unit cubes, and $(m3^d - \psi(m) 2^d)$ one-unit cubes for a total of

$2^d+(3^d-1)m-(2^d-1)\,\psi(m)$ cubes in the entire dissection. This dissection has $(2^d-1)\,\psi(m)$ fewer cubes than the simple dissection. For $m\in\{0, 1, ..., 2^d-2\}$, $2^d+(3^d-1)m-(2^d-1)\,\psi(m)<6^d$. Also, for each $i\in\{0, 1, ..., 2^d-2\}$, there is some m such that $2^d+(3^d-1)m-(2^d-1)\,\psi(m)=1 (\bmod 2^d-1)$ since $\gcd(2^d-1, 3^d-1)=1$. Therefore, from Theorem 1, we may conclude that $c(d)<6^d$. Computer experiments show that for $d\leqslant 25$ that in the case $\gcd(2^d-1, 3^d-1)=1$ that $c(d)<6^d/4.1$.

3.2. *General Asymptotics*

THEOREM 2. *If* $\gcd(2^d-1, k^d-1)=1$ *for infinitely many values of* d, *then* $c(d)=O((2k)^{d-1})$.

If k is even, then $\gcd(2^d-1, k^d-1)=\gcd(2^d-1, (k/2)^d-1)$, so we assume k is odd. We generate each element a_i in A by first subdividing Q_d into 2^d cubes and then subdividing i of these cubes into k^d smaller cubes. At this point, we have subdivided Q_d into $2^d+i(k^d-1)$ cubes. Suppose $i=\sum_{p=1}^{m} 2^{n_p}$. Then we choose our i cubes in $2^{n_p}\times 1^{d-n_p}$ slabs. For each slab, we then recoalesce groups of 2^d small cubes into cubes having twice the edge length. In the "n_p" slab, we may recoalesce $(2k)^{n_p}\,(k-1)^{d-n_p}$ small cubes in this manner, leaving $(2k)^{n_p}\,(k^{d-n_p}-(k-1)^{d-n_p})$ small cubes unmolested. Then if we compute the number of small cubes in all of Q_d, we obtain

$$\begin{aligned}
\#\text{ of small cubes} &= \sum_{p=1}^{m} (2k)^{n_p}\,(k^{d-n_p}-(k-1)^{d-n_p})\\
&\leqslant \sum_{n=0}^{d-1} (2k)^{n}\,(k^{d-n}-(k-1)^{d-n})\\
&= \sum_{n=0}^{d-1} (2k)^{n}\left(\sum_{q=0}^{d-n-1} k^q(k-1)^{d-n-1-q}\right)\\
&\leqslant \sum_{n=0}^{d-1} (2k)^{n}\,(d-n-1)k^{d-n-1})\\
&\leqslant \sum_{n=0}^{d-1} (2k)^{d-1}\,(d-n-1)2^{-(d-n-1)}\\
&\leqslant (2k)^{d-1}\sum_{n=0}^{d-1} n2^{-n}\\
&\leqslant (2k)^{d-1}\sum_{n=0}^{\infty} n2^{-n}\\
&= 4(2k)^{d-1}
\end{aligned}$$

which is an $O((2k)^{d-1})$ quantity. The number of recoalesced cubes is bounded by k^d and the number of cubes into which Q_d was originally decomposed is bounded by 2^d. It follows that for each i, we can choose $a_i \in A$ such that

$$a_i \leqslant \left(4 + \frac{k}{2^{d-1}} + \frac{2}{k^{d-1}}\right)(2k)^{d-1}.$$

This last quantity is $O((2k)^{d-1})$ and independent of i. This implies $c(d) = O((2k)^{d-1})$ as desired. ∎

We now prove our general asymptotic bound for all d, namely that $c(d) = O((2d)^{d-1})$. In order to do this we need two lemmas, the first of which appears in Erdős [2] and the second of which is an application of elementary calculus.

Lemma 1. *For any positive integer d,* $\gcd(2^d - 1, 3^d - 1, ..., (d+1)^d - 1) = 1$.

The following proof appears in Erdős [2]. Suppose this statement is not true. Then there is some prime number p which is a factor of $k^d - 1$ for each $k \in A = \{1, 2, 3, ..., d+1\}$. We then note that $k^d \equiv 1 \bmod p$ for such integers k. Further, we note that $p > (d+1)$, since if not then p is a factor of $p^d - 1$ which is absurd. Therefore, A is a set of $d+1$ distinct elements of the group $\mathbf{Z}_p^+$ under multiplication, all of whose orders divide d. Since this group is cyclic, there can be at most d such elements which is a contradiction. ∎

Lemma . *There is some constant C such that $S(m) = \sum_{k=1}^{m-1} (1 - k/m)^{m-2} < C$ for all integers $m > 2$.*

The kth term of $S(m)$ has limit e^{-k} as m increases without bound, so each term has some constant c_k which bounds it. What we do not know is whether we can find a set of such constants $\{c_1, c_2, ...\}$ such that the series $\sum c_k$ converges. We show that, for $k \geqslant 4$, e^{-k} in fact bounds $(1 - k/m)^{m-2}$ from above, and so we may choose

$$\begin{aligned} C &= c_1 + c_2 + c_3 + \sum_{k=4}^{\infty} e^{-k} \\ &= c_1 + c_2 + c_3 + e^{-4}(1 - e^{-1})^{-1}. \end{aligned}$$

For $k \geqslant 4$, we consider the function $f(x) = (1 - k/x)^{x-2}$. We seek to show that $f(x)$ is increasing for $x \geqslant k+1$. This will suffice to show that, on $[k+1, \infty)$, $f(x)$ is bounded by $\lim_{x \to \infty} f(x) = e^{-k}$. We compute $f'(x) = (1 - k/x)^{x-3}((1 - k/x)\ln(1 - k/x) + (x-2)k/x^2)$ which we wish to show is positive for $x \geqslant k+1$. Since $1-k/x>0$ for $x \geqslant k+1$, it is enough to show that $g(x)=\ln(1-k/x)+k(x-2)/(x^2(1-k/x))>0$. Expanding each term of $g(x)$ in a power series of $1/x$, we obtain

$$\ln(1-k/x)=\sum_{i=1}^{\infty} -(k^i)/(ix^i)$$

$$kx/x^2(1-k/x)=\sum_{i=1}^{\infty} k^i/x^i$$

$$2k/x^2(1-k/x)=\sum_{i=2}^{\infty} 2k^{i-1}/x^i$$

and, since these series all converge absolutely when $x \geqslant k+1$, we have that

$$g(x)=\sum_{i=2}^{\infty} \frac{(i-1)k^i-2ik^{i-1}}{ix^i}.$$

The numerator of each term is $k^{i-1}(k(i-1)-2i)$. Since $k \geqslant 4$ and $i \geqslant 2$, this expression is at least $4^{i-1}(4i-4-2i)=4^{i-1}(2i-4) \geqslant 0$ with equality only when $i=2$. Therefore, $g(x)$ is positive, and then so is $f'(x)$. Hence, $f(x)$ is increasing for $x \in [k+1, \infty)$. This completes the proof of the lemma. ∎

THEOREM 3. $c(d)=O((2d)^{d-1})$.

We begin by subdividing Q_d into 2^d mini-cubes. Since $2^d \geqslant d-1$ for all positive d, we may subdivide $d-1$ of the mini-cubes each into 2^d micro-cubes. Now, from Lemma 1, given an $a \in \{0, 1, ..., 2^d-2\}$, there is some $(d-1)$-tuple $(a_3, a_4, ..., a_{d+1})$ such that $0 \leqslant a_i \leqslant 2^d-2$ and such that $1+\sum_{k=3}^{d+1} a_k(k^d-1) \equiv a (\text{mod } 2^d-1)$. With this in mind, for each k, we subdivide a_k of the micro-cubes in the $(k-2)$nd mini-cube into k^d nano-cubes. At this point, the entire cube Q_d has been subdivided into 2^d-d+1 mini-cubes, $(d-1)\,2^d-\sum_{k=3}^{d+1} a_k$ micro–cubes, and $\sum_{k=3}^{d+1} a_k k^d$ nano-cubes. Summing these, we have that Q_d is subdivided into

$$1+(d-1)(2^d-1)+\sum_{k=3}^{d+1} a_k(k^d-1)$$

smaller cubes. This last quantity is equivalent to a modulo 2^d-1.

Now, if the $a_k=\sum_{p=1}^{m} 2^{n_p}$ nano–cubes in the $(k-2)$nd micro-cube are arranged in $2^{n_p} \times 1^{d-n_p}$ slabs, as in the proof of Theorem 2, then we may recoalesce many of these nano-cubes into cubes having twice the edge length. By the computation in Theorem 2, there will be at most $4(2k)^{d-1}$ nano-cubes in the $(k-2)$nd micro-cube remaining after the recoalescing process. Furthermore, the total number of cubes into which the $(k-2)$nd micro-cube has been subdivided satisfies

$$\text{number of cubes} \leqslant \left(4+\frac{d+1}{2^{d-1}}+\frac{2}{3^{d-1}}\right)(2k)^{d-1} \leqslant \frac{37}{6}(2k)^{d-1}.$$

This follows from the last formula in the proof of Theorem 2, noting that $3 \leqslant k \leqslant d+1$.

It follows that the total number of cubes into which Q_d has been subdivided satisfies

$$\begin{aligned}
\#(\text{cubes}) &\leqslant 1+(d-1)(2^d-1)+\sum_{k=3}^{d+1}\frac{37}{6}(2k)^{d-1} \\
&\leqslant 1+(d-1)(2^d-1)+\frac{37}{6}(2(d+1))^{d-1}\sum_{k=3}^{d+1}\left(\frac{k}{d+1}\right)^{d-1} \\
&\leqslant (2d)^{d-1}\left((2d)^{1-d}+2d^{2-d}+\frac{37}{6}eC\right),
\end{aligned}$$

where C is the constant in Lemma 2 bounding $S(d+1)$ which is greater than $\sum_{k=3}^{d+1}(k/(d+1))^{d-1}$. This last quantity is $O((2d)^{d-1})$, which proves the result. ▌

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